so $m+1$ has to be a perfect square. So there is no Shoesmith labelling for K_7 or K_{11}. Now K_3 (= C_3) does have a Shoesmith labelling. But what about K_{15}?

Case (ii) $n = 4m$. Here $\frac{1}{2} n(n+1) = 2m(4m+1)$ and we require $a+b = 4m$ and $ab+b = 4m^2 + m$. Thus we require $(a+1)(4m-a) = 4m^2 + m$, i.e.

$$a^2 - (4m-1)a + 4m^2 - 3m = 0.$$

The discriminant is $4m+1$, so we require $4m+1$ to be a perfect square. So no Shoesmith labelling exists for $n = 4$ or 12 or 16 or 20. Now Shoesmith has shown recently that a there is a labelling for K_8 : label the vertices by 1,3,7,11,13,23,28 and 32. But what about K_{24}?

Case (iii) $n = 4m + 1$. Here $\frac{1}{2}n(n+1) = 2(4m^2 + 3m) + 1$ and we require

$$a+b = 4m+1 \quad \text{and} \quad ab+b = 4m^2 + 3m+1.$$

This leads to $(a-2m)^2 = m$, so m must be a perfect square. For $m=1$ ($n=5$) there are solutions with $a=1$ and with $a=3$, but there are no labellings for K_9 or K_{13}. What about K_{17}?

Case (iv) $n = 4m+2$. Here $\frac{1}{2} n(n+1) = 2(4m^2 + 5m + 1) + 1$, so we require

$$a+b = 4m+2 \quad \text{and} \quad ab+b = 4m^2 + 5m + 2,$$

This leads to $a^2 - (4m+1)\, a + 4m^2 + m = 0$, and, as in case (ii), this requires $4m+1$ (= $n-1$) to be a perfect square. So no Shoesmith labelling exists for $n = 6$ or 14 or 18 or 22. What about K_{10}? This is the smallest unknown case.

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NOTE ON THE SECURE-DOMINATION NUMBER OF A GRAPH

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ABSTRACT. It is shown that the ratio of the secure-domination number of a graph to its order can be bounded away from $\frac{1}{2}$ in families including arbitrarily large connected graphs. Infinitely many trees and infinitely many connected graphs of girth 6 are found in which the ratio is $\frac{4}{7}$.

1. Introduction

All graphs in this note will be finite and simple. The vertex set of a graph G will be denoted $V(G)$, and its edge set will be denoted $E(G)$. An edge incident to $u, v \in V(G), u \neq v$, may be denoted uv or vu. If $uv \in E(G)$, then u and v are adjacent in G. If $v \in V(G)$, its open and closed neighbor sets in G are, respectively, $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. When the graph of reference is clear from the context, the subscript in this notation may be omitted. For instance, if $S \subseteq V(G)$ we can define $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Suppose that $\emptyset \neq S \subseteq V(G)$. We summarize definitions originally given in [2]:

In an attack on S, each vertex in $N(S) \backslash S$ attacks exactly one of its neighbors in S; in a defense of S, each vertex in S defends either itself or one of its neighbors in S; given an attack on S, and a defense of S, the defense defends against (or thwarts, or defeats) the attack if and only if each $v \in S$ has at least as many defenders as attackers of S. If for every attack on S there is a defense of S against that attack, then S is said to be *secure* (in G).

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One easy way to see that a set $S \subseteq V(G)$ is not secure is to locate $v \in S$ with more potential attackers than potential defenders:

$$|N(v)\backslash S| > |N[v] \cap S|.$$

For, supposing this inequality holds, if we let every vertex of $N(v)\backslash S$ attack v, then however the attacks on S of the other vertices of $N(S)\backslash S$ are assigned, and however the defensive capabilities in S are deployed, v will have more attackers than defenders. This observation is a special case of a more general one: if, for some $X \subseteq S$, $|N(X)\backslash S| > |N[X] \cap S|$, then S is not secure. The fundamental theorem of secure-set theory ([2] and [4]) asserts that the converse holds: if $|N(X)\backslash S| \leq |N[X] \cap S|$ for all $X \subseteq S$, then S is secure.

$S \subset V(G)$ is *dominating* in G if and only if $N[S] = V(G)$. If S is both dominating and secure, we will say that S is a secure-dominating set. The secure-domination number of G is $\gamma_s(G) = \min\{|S| : S \subseteq V(G)$ is a secure-dominating set in $G\}$; $\gamma_s(G)$ is well-defined because $V(G)$ is clearly secure-dominating in G.

Since a secure-dominating set is a set which is both secure and dominating, there is no grammatical nor logical reason for the hyphen between "secure" and "dominating". But there is a historical/cultural reason for the hyphen, which we are grateful to the referee for alerting us to. Recently there has been an eruption of interest in "secure domination", meaning something quite different from our "secure-domination". A "secure dominating set" is a dominating set which is not necessarily secure in the sense defined here; rather, the set's property of being a dominating set is "secure" in that the set can be perturbed in certain ways, with the result guaranteed to be dominating. See [3]. The reader is also warned that the common notation for the "secure domination number" of a graph G (guess the definition!) is $\gamma_s(G)$, the same notation as that used here for the secure-domination number of G. For now, we will just take note and endure the ambiguity.

As noted in [5], if $S \subseteq V(G)$ is dominating and $|S| < \frac{|V(G)|}{2}$, then

$$\begin{aligned}|N(S)\backslash S| &= |V(G)\backslash S| \\ &= |V(G)| - |S| \\ &> \frac{|V(G)|}{2} \\ &> |S|\end{aligned}$$

Therefore, S cannot be secure. Consequently, $\gamma_s(G) \geq \lceil \frac{|V(G)|}{2} \rceil$.

In [5] $\gamma_s(G)$ is determined for all graphs G among the "usual suspects" – paths, cycles, cubes, complete multipartite graphs, and the Petersen graph– and it turns out that $\gamma_s(G) = \lceil \frac{|V(G)|}{2} \rceil$ for all such G except $G = C_n$, $n \equiv 2 \bmod 4$, $n \geq 6$. In these cases $\gamma_s(G) = \frac{n}{2} + 1$. Another result in [5] and recent unpublished work [1] shows that $\gamma_s = \lceil \frac{|V(G)|}{2} \rceil$ for a great many graphs G. Which raises the question: How much larger than $\lceil \frac{|V(G)|}{2} \rceil$ can $\gamma_s(G)$ be? To pose the question more precisely, how much larger than $\frac{1}{2}$ can $\frac{\gamma_s(G)}{|V(G)|}$ be? This ratio is $\frac{2}{3}$ for $G \in \{P_3, K_3, C_6\}$ (P_3 is the path on 3 vertices, K_3 is the complete graph on 3 vertices, and C_6 the cycle on 6 vertices). Taking disjoint unions (or "sums") of these graphs gives arbitrarily large graphs with the same ratio, but this is a very unsatisfying observation which leads to the question: Is there a number $\eta > \frac{1}{2}$ such that for infinitely many connected graphs G, $\frac{\gamma_s(G)}{|V(G)|} \geq \eta$, and, if so, what is the largest (if any) η that will satisfy this statement? We could call the largest η (or, the supremum of all the η that satisfy the statement) $\limsup_{G \text{ connected}} \frac{\gamma_s(G)}{|V(G)|}$, borrowing a term from analysis.

The class of connected graphs can be replaced by other classes. By previous remarks we have, trivially,

$$\limsup_{G \text{ simple}} \frac{\gamma_s(G)}{|V(G)|} \geq \frac{2}{3}.$$

From [5] it can be concluded that

$$\limsup_{G \in \mathcal{G}} \frac{\gamma_s(G)}{|V(G)|} = \frac{1}{2}$$

for many infinite classes $\mathcal{G}$ of graphs – not only paths, cycles, and complete multipartite graphs, but also any infinite class $\mathcal{G}$ contained in the union of these classes. Recent work [1] extends this result to grids (Cartesian products of paths), cylindrical grids (paths and cycles), and toroidal grids (cycles and cycles).

In the main results of this note, in the next section, we show that

$$\limsup_{G \in \mathcal{G}} \frac{\gamma_s(G)}{|V(G)|} \geq \frac{4}{7}$$

for $\mathcal{G} \in \{\{\text{trees}\}, \{\text{connected graphs with girth 6}\}\}$.

2. Results

Let $G + H$ denote the disjoint union, or sum, of two graphs G and H. Let $\underbrace{G + G + \cdots + G}_{m-times}$ be denoted by mG.

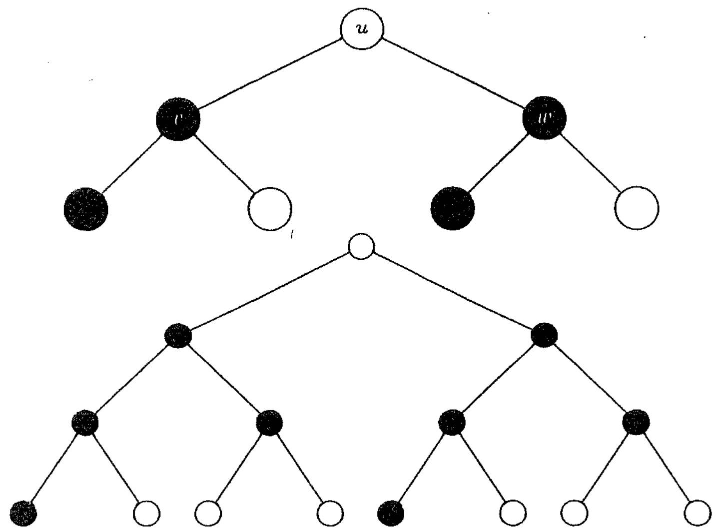

FIGURE 1. G_3 and G_4, with minimum secure-dominating sets.

For each positive integer n, let G_n denote the full binary tree with 2^{n-1} leafs (terminal nodes), each a distance $n-1$ in G_n from the root of the tree (which, for $n > 1$, is the only vertex of degree 2 in the graph). For $n > 1$, G_n is formed by making the root of G_n adjacent to the roots of two copies of G_{n-1}. Note that $G_1 = K_1$ and $G_2 = P_3$, which has secure-domination number 2. In Figure 1 are G_3 and G_4, with a secure-dominating set of vertices indicated by blacking the members of the set. (In fact, two different minimum secure-dominating sets for G_3 are displayed, one of them in two G_3's in the G_4.) Three vertices of G_3 are labeled for future reference.

Since $\gamma_s(H) \geq \lceil \frac{|V(G)|}{2} \rceil$ for any graph H, this picture shows that $\gamma_s(G_4) = 8$ and $\gamma_s(G_3) = 4$.

We believe that

$$\lim_{n\to\infty} \frac{\gamma_s(G_n)}{|V(G_n)|} = \lim_{n\to\infty} \frac{\gamma_s(G)}{2^n - 1}$$

exists and is greater that $\frac{1}{2}$ – somewhere in the interval $(\frac{1}{2}, \frac{9}{16})$ – but we have not been able to prove this, nor even the corresponding claim for the subsequence $G_{4n}, n = 1, 2, \ldots$. However, we can achieve the goal of showing that $\limsup_{G \text{ is a tree}} \frac{\gamma_s(G)}{|V(G)|} > \frac{1}{2}$ without estimating $\gamma_s(G_n)$, by using the properties of G_3 alone.

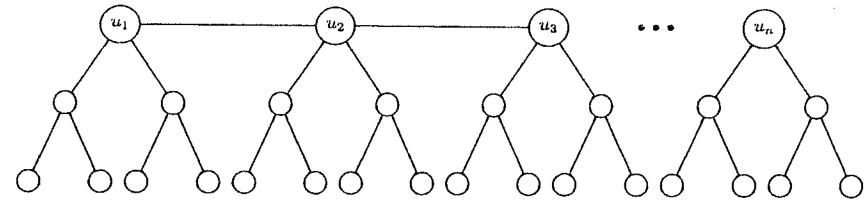

FIGURE 2. H_n

Let H_n denote the graph obtained by making the root vertices of n G_3's the vertices of a path. See Figure 2.

Theorem 1. $\gamma_s(H_n) = 4n$

Proof. Let the vertices along the path in the definition of H_n be $u_1, \ldots, u_n$, let the G_3 of which u_i is the root be denoted $G^{(i)}$, and let v_i and w_i play the roles in $G^{(i)}$ that v and w play in Figure 1.

By copying the secure-dominating set depicted in Figure 1 (G_3) on each $G^{(i)}$, we obtain a secure-dominating set in H_n with $4n$ vertices. To finish the proof, it suffices to show that if $S \subseteq V(H_n)$ is a secure-dominating set in H_n, then $|S \cap V(G^{(i)})| \geq 4$, $i = 1, \ldots, n$.

Suppose that S is such a set, and let $S_i = S \cap V(G^{(i)})$. If S_i is secure and dominating in $G^{(i)}$, then $|S_i| \geq \gamma_s(G_3) = 4$.

Case 1 $u_i \notin S$:
Since S_i cannot be attacked from outside $G^{(i)}$, in this case, S_i must be secure in $G^{(i)}$, but not necessarily dominating. However, S_i is dominating in $G^{(i)} - u_i$. The only way S_i can fail to be dominating in $G^{(i)}$ is if $v_i, w_i \notin S_i$. But if this were so, then S_i could be dominating in $G^{(i)} - u_i$ only if all 4 leafs (terminal vertices) in $G^{(i)}$ are in S_i. Thus, in any case, $|S_i| \geq 4$.

Case 2 $u_i \in S$:
In this case S_i is dominating in $G^{(i)}$, but may fail to be secure in $G^{(i)}$. However, if $|S_i| \leq 3$ and $u_i \in S_i$, then S_i is dominating in $G^{(i)}$ only if $S_i = \{u_i, v_i, w_i\}$, in which case $X = \{v_i, w_i\}$ has 4 attackers and $|N_{H_n}[X] \cap S| = 3$, contradicting the security of S. Consequently, $|S_i| \geq 4$.

□

Let X_n denote the graph obtained by making each vertex on a path $P \simeq P_n$ adjacent to one vertex of a C_6. See Figure 3.

Theorem 2. $\gamma_s(X_n) = 4n$.

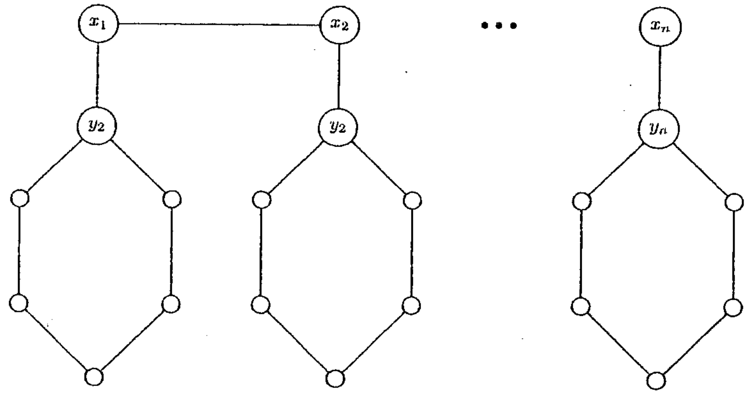

FIGURE 3. X_n

Proof. Let the vertices along the path P be $x_1, \ldots, x_n$ and let the neighbor of x_i on the C_6 allotted to x_i be y_i. Let H_i be the subgraph of X_n induced by x_i with the vertices of the C_6 containing y_i.

The set made up of $x_1, \ldots, x_n$ together with the 3 vertices furthest from y_i in H_i, for each $i = 1, \ldots, n$, is clearly secure and dominating, with $4n$ vertices. So $\gamma_s(X_n) \leq 4n$. Suppose that $S \subseteq V(X_n)$ is secure and dominating in X_n. The proof is over if we show that $|S \cap V(H_i)| \geq 4$, $i = 1, \ldots, n$.

Let $S_i = S \cap V(H_i)$ and suppose that $|S_i| \leq 3$ for some i.

Case 1 $x_i \notin S_i$:
Then S_i must be both dominating and secure in the 6-cycle $H_i - x_i$. But it is straightforward to see that this is impossible for a set of 3 or fewer vertices.

Case 2 $x_i \in S_i$:
Then $|S_i \backslash \{x_i\}| \leq 2$, and S_i is dominating in H_i. The only ways to achieve the latter by picking 2 vertices of $C_6 \simeq H_i - x_i$ to go with x_i in forming S_i involve choosing two vertices on the cycle that are not adjacent. But then S cannot be secure, because at least one of these vertices is of degree 2 with both of its neighbors not in S. □

3. Problems remaining

(1) Find $\limsup\limits_{n \to \infty} \frac{\gamma_s(G_n)}{2^n - 1}$. We are pretty sure that this lim sup is a lim, but we could be wrong.

(2) Find $\limsup\limits_{G \in \mathcal{G}} \frac{\gamma_s(G)}{|V(G)|}$ for $\mathcal{G} \in \{\{\text{trees}\}, \{\text{connected graphs}\}\}$. We expect that, at least, improvements on our lower estimate of $\frac{4}{7}$ will be found.

(3) Does there exist a connected graph G on more than one vertex such that $\frac{\gamma_s(G)}{|V(G)|} > \frac{2}{3}$?

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